9.4 Perform Rotations

Before You rotated figures about the origin.

Now You will rotate figures about a point.

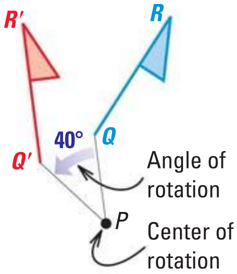
Why? So you can classify transformations, as in Exs. 3–5.

Key Vocabulary
- **center of rotation**
- **angle of rotation**
- **rotation,** *p. 272*

Recall from Lesson 4.8 that a *rotation* is a transformation in which a figure is turned about a fixed point called the **center of rotation**. Rays drawn from the center of rotation to a point and its image form the **angle of rotation**.

A rotation about a point P through an angle of $x°$ maps every point Q in the plane to a point Q' so that one of the following properties is true:

- If Q is not the center of rotation P, then $QP = Q'P$ and $m\angle QPQ' = x°$, or
- If Q is the center of rotation P, then the image of Q is Q.



A 40° counterclockwise rotation is shown at the right. Rotations can be *clockwise* or *counterclockwise*. In this chapter, all rotations are counterclockwise.

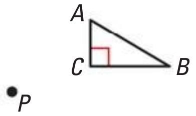
DIRECTION OF ROTATION

clockwise

counterclockwise

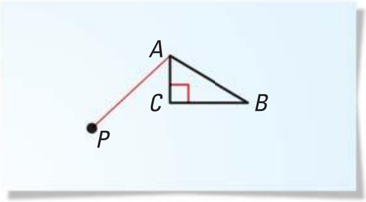
EXAMPLE 1 Draw a rotation

Draw a 120° rotation of $\triangle ABC$ about P.

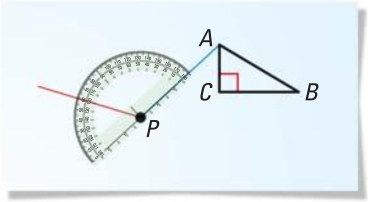


Solution

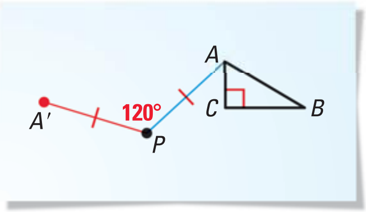
STEP 1 **Draw** a segment from A to P.



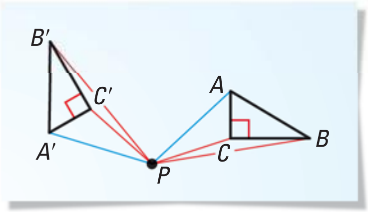
STEP 2 **Draw** a ray to form a 120° angle with $\overline{PA}$.



STEP 3 **Draw** A' so that $PA' = PA$.

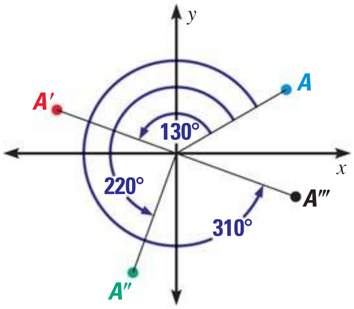


STEP 4 **Repeat** Steps 1–3 for each vertex. Draw $\triangle A'B'C'$.



ROTATIONS ABOUT THE ORIGIN You can rotate a figure more than 180°. The diagram shows rotations of point A 130°, 220°, and 310° about the origin. A rotation of 360° returns a figure to its original coordinates.

There are coordinate rules that can be used to find the coordinates of a point after rotations of 90°, 180°, or 270° about the origin.



USE ROTATIONS

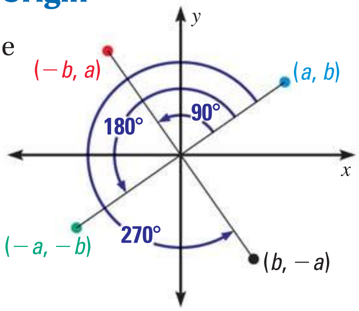
You can rotate a figure more than 360°. However, the effect is the same as rotating the figure by the angle minus 360°.

KEY CONCEPT — *For Your Notebook*

Coordinate Rules for Rotations about the Origin

When a point (a, b) is rotated counterclockwise about the origin, the following are true:

1. For a rotation of 90°, $(a, b) \to (-b, a)$.
2. For a rotation of 180°, $(a, b) \to (-a, -b)$.
3. For a rotation of 270°, $(a, b) \to (b, -a)$.



EXAMPLE 2 Rotate a figure using the coordinate rules

Graph quadrilateral $RSTU$ with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$. Then rotate the quadrilateral 270° about the origin.

Solution

Graph $RSTU$. Use the coordinate rule for a 270° rotation to find the images of the vertices.

ANOTHER WAY

For an alternative method for solving the problem in Example 2, turn to page 606 for the **Problem Solving Workshop**.

$$(a, b) \to (b, -a)$$

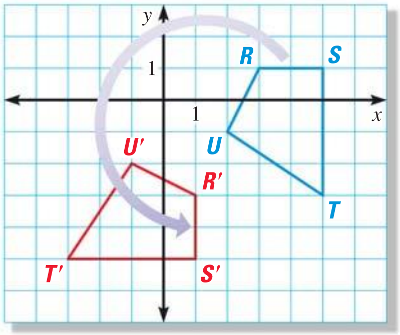
$R(3, 1) \to R'(1, -3)$

$S(5, 1) \to S'(1, -5)$

$T(5, -3) \to T'(-3, -5)$

$U(2, -1) \to U'(-1, -2)$

Graph the image $R'S'T'U'$.



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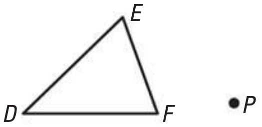
✓ GUIDED PRACTICE for Examples 1 and 2

1. Trace △DEF and P. Then draw a 50° rotation of △DEF about P.

2. Graph △JKL with vertices $J(3, 0)$, $K(4, 3)$, and $L(6, 0)$. Rotate the triangle 90° about the origin.



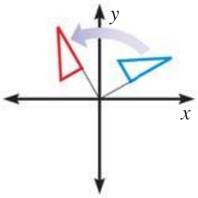
USING MATRICES You can find certain images of a polygon rotated about the origin using matrix multiplication. Write the rotation matrix to the left of the polygon matrix, then multiply.

KEY CONCEPT *For Your Notebook*

Rotation Matrices (Counterclockwise)

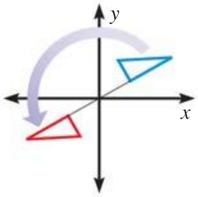
90° rotation

$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$



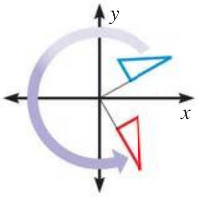
180° rotation

$$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$



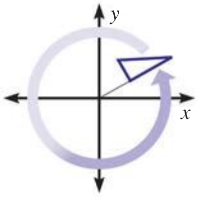
270° rotation

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$



360° rotation

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$



READ VOCABULARY

Notice that a 360° rotation returns the figure to its original position. Multiplying by the matrix that represents this rotation gives you the polygon matrix you started with, which is why it is also called the *identity matrix*.

EXAMPLE 3 Use matrices to rotate a figure

Trapezoid *EFGH* has vertices *E*(−3, 2), *F*(−3, 4), *G*(1, 4), and *H*(2, 2). Find the image matrix for a 180° rotation of *EFGH* about the origin. Graph *EFGH* and its image.

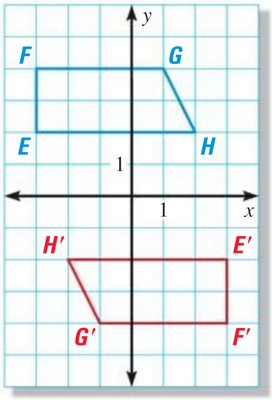
Solution

STEP 1 **Write** the polygon matrix: $\begin{array}{c}\begin{matrix}E & F & G & H\end{matrix}\\ \begin{bmatrix}-3 & -3 & 1 & 2\\2 & 4 & 4 & 2\end{bmatrix}\end{array}$

STEP 2 **Multiply** by the matrix for a 180° rotation.

$$\underset{\text{Rotation matrix}}{\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}}\underset{\text{Polygon matrix}}{\begin{array}{c}\begin{matrix}E & F & G & H\end{matrix}\\ \begin{bmatrix}-3 & -3 & 1 & 2\\2 & 4 & 4 & 2\end{bmatrix}\end{array}} = \underset{\text{Image matrix}}{\begin{array}{c}\begin{matrix}E' & F' & G' & H'\end{matrix}\\ \begin{bmatrix}3 & 3 & -1 & -2\\-2 & -4 & -4 & -2\end{bmatrix}\end{array}}$$

STEP 3 **Graph** the preimage *EFGH*.
Graph the image *E′F′G′H′*.



AVOID ERRORS

Because matrix multiplication is not commutative, you should always write the rotation matrix first, then the polygon matrix.

GUIDED PRACTICE for Example 3

Use the quadrilateral *EFGH* in Example 3. Find the image matrix after the rotation about the origin. Graph the image.

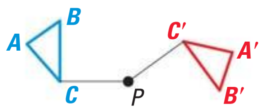
3. 90° **4.** 270° **5.** 360°

THEOREM *For Your Notebook*

THEOREM 9.3 Rotation Theorem

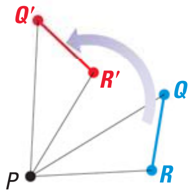
A rotation is an isometry.

Proof: Exs. 33–35, p. 604

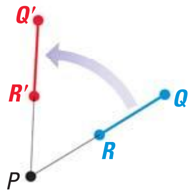


$\triangle ABC \cong \triangle A'B'C'$

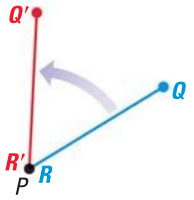
CASES OF THEOREM 9.3 To prove the Rotation Theorem, you need to show that a rotation preserves the length of a segment. Consider a segment $\overline{QR}$ rotated about point P to produce $\overline{Q'R'}$. There are three cases to prove:



Case 1 R, Q, and P are noncollinear.



Case 2 R, Q, and P are collinear.



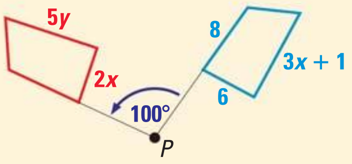
Case 3 P and R are the same point.

EXAMPLE 4 Standardized Test Practice

The quadrilateral is rotated about P. What is the value of y?

Ⓐ $\frac{8}{5}$ Ⓑ 2

Ⓒ 3 Ⓓ 10



Solution

By Theorem 9.3, the rotation is an isometry, so corresponding side lengths are equal. Then $2x = 6$, so $x = 3$. Now set up an equation to solve for y.

$5y = 3x + 1$ **Corresponding lengths in an isometry are equal.**

$5y = 3(3) + 1$ **Substitute 3 for x.**

$y = 2$ **Solve for y.**

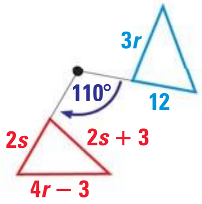
▶ The correct answer is B. Ⓐ Ⓑ Ⓒ Ⓓ

✓ GUIDED PRACTICE for Example 4

6. Find the value of r in the rotation of the triangle.

Ⓐ 3 Ⓑ 5

Ⓒ 6 Ⓓ 15



9.4 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 13, 15, and 29

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 20, 21, 23, 24, and 37

SKILL PRACTICE

1. **VOCABULARY** What is a *center of rotation*?

2. ★ **WRITING** *Compare* the coordinate rules and the rotation matrices for a rotation of 90°.

EXAMPLE 1
on p. 598
for Exs. 3–11

IDENTIFYING TRANSFORMATIONS **Identify the type of transformation, *translation*, *reflection*, or *rotation*, in the photo. *Explain* your reasoning.**

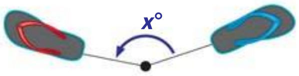
3.

4.

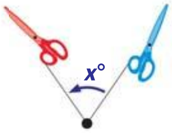
5.

ANGLE OF ROTATION **Match the diagram with the angle of rotation.**

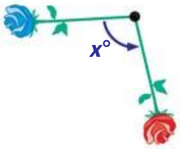
6.



7.



8.



A. 70°

B. 100°

C. 150°

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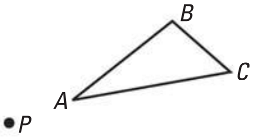
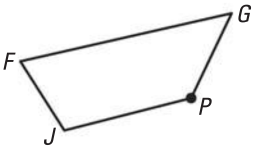
ROTATING A FIGURE **Trace the polygon and point *P* on paper. Then draw a rotation of the polygon the given number of degrees about *P*.**

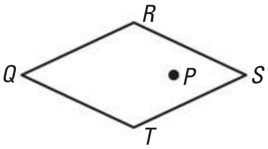
9. 30°



10. 150°



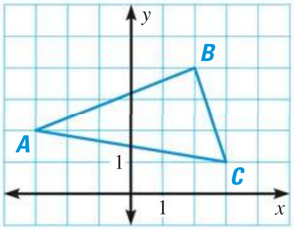
11. 130°



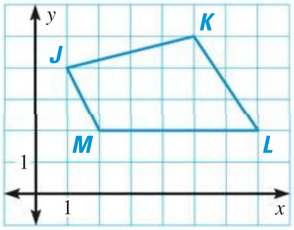
EXAMPLE 2
on p. 599
for Exs. 12–14

USING COORDINATE RULES **Rotate the figure the given number of degrees about the origin. List the coordinates of the vertices of the image.**

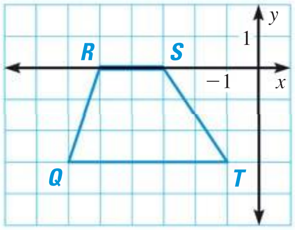
12. 90°



13. 180°



14. 270°



EXAMPLE 3
on p. 600
for Exs. 15–19

USING MATRICES **Find the image matrix that represents the rotation of the polygon about the origin. Then graph the polygon and its image.**

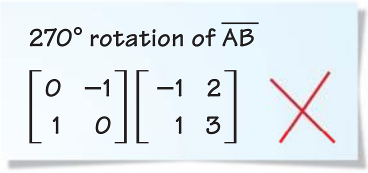
15. $\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 1 & 5 & 4 \\ 4 & 6 & 3 \end{bmatrix} \end{array}$; 90°

16. $\begin{array}{c} \begin{matrix} J & K & L \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -3 \end{bmatrix} \end{array}$; 180°

17. $\begin{array}{c} \begin{matrix} P & Q & R & S \end{matrix} \\ \begin{bmatrix} -4 & 2 & 2 & -4 \\ -4 & -2 & -5 & -7 \end{bmatrix} \end{array}$; 270°

ERROR ANALYSIS **The endpoints of $\overline{AB}$ are $A(-1, 1)$ and $B(2, 3)$. *Describe* and correct the error in setting up the matrix multiplication for a 270° rotation about the origin.**

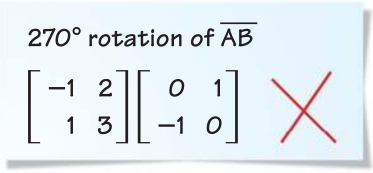
18.



270° rotation of $\overline{AB}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

19.



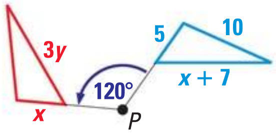
270° rotation of $\overline{AB}$

$\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

EXAMPLE 4
on p. 601
for Exs. 20–21

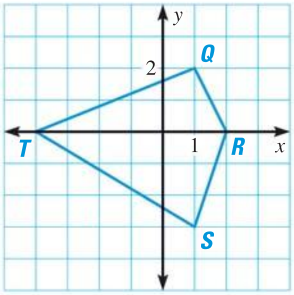
20. ★ **MULTIPLE CHOICE** What is the value of y in the rotation of the triangle about P?

Ⓐ 4 Ⓑ 5 Ⓒ $\frac{17}{3}$ Ⓓ 10



21. ★ **MULTIPLE CHOICE** Suppose quadrilateral $QRST$ is rotated 180° about the origin. In which quadrant is Q'?

Ⓐ I Ⓑ II Ⓒ III Ⓓ IV



22. **FINDING A PATTERN** The vertices of $\triangle ABC$ are $A(2, 0)$, $B(3, 4)$, and $C(5, 2)$. Make a table to show the vertices of each image after a 90°, 180°, 270°, 360°, 450°, 540°, 630°, and 720° rotation. What would be the coordinates of A' after a rotation of 1890°? *Explain.*

23. ★ **MULTIPLE CHOICE** A rectangle has vertices at (4, 0), (4, 2), (7, 0), and (7, 2). Which image has a vertex at the origin?

Ⓐ Translation right 4 units and down 2 units

Ⓑ Rotation of 180° about the origin

Ⓒ Reflection in the line $x = 4$

Ⓓ Rotation of 180° about the point (2, 0)

24. ★ **SHORT RESPONSE** Rotate the triangle in Exercise 12 90° about the origin. Show that corresponding sides of the preimage and image are perpendicular. *Explain.*

25. **VISUAL REASONING** A point in space has three coordinates (x, y, z). What is the image of point (3, 2, 0) rotated 180° about the origin in the xz-plane? (*See* Exercise 30, page 585.)

CHALLENGE **Rotate the line the given number of degrees (a) about the x-intercept and (b) about the y-intercept. Write the equation of each image.**

26. $y = 2x - 3$; 90°

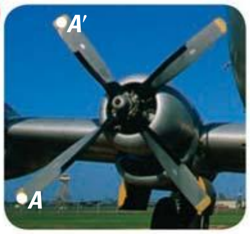
27. $y = -x + 8$; 180°

28. $y = \frac{1}{2}x + 5$; 270°

PROBLEM SOLVING

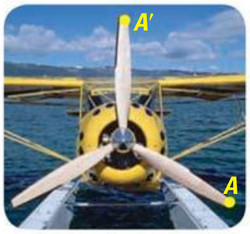
ANGLE OF ROTATION **Use the photo to find the angle of rotation that maps *A* onto *A*′. *Explain* your reasoning.**

29.



30.

31.



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32. **REVOLVING DOOR** You enter a revolving door and rotate the door 180°. What does this mean in the context of the situation? Now, suppose you enter a revolving door and rotate the door 360°. What does this mean in the context of the situation? *Explain*.

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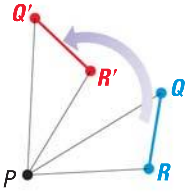


33. **PROVING THEOREM 9.3** Copy and complete the proof of Case 1.

Case 1 The segment is noncollinear with the center of rotation.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'.

PROVE ▶ $QR = Q'R'$

STATEMENTS	REASONS
1. $PQ = PQ'$, $PR = PR'$, $m\angle QPQ' = m\angle RPR'$	**1.** Definition of ?
2. $m\angle QPQ' = m\angle QPR' + m\angle R'PQ'$ $m\angle RPR' = m\angle RPQ + m\angle QPR'$	**2.** ?
3. $m\angle QPR' + m\angle R'PQ' = m\angle RPQ + m\angle QPR'$	**3.** ? Property of Equality
4. $m\angle QPR = m\angle Q'PR'$	**4.** ? Property of Equality
5. ? $\cong$?	**5.** SAS Congruence Postulate
6. $\overline{QR} \cong \overline{Q'R'}$	**6.** ?
7. $QR = Q'R'$	**7.** ?

PROVING THEOREM 9.3 **Write a proof for Case 2 and Case 3. (Refer to the diagrams on page 601.)**

34. **Case 2** The segment is collinear with the center of rotation.

GIVEN ▶ A rotation about P maps Q to Q' and R to R'.
P, Q, and R are collinear.

PROVE ▶ $QR = Q'R'$

35. **Case 3** The center of rotation is one endpoint of the segment.

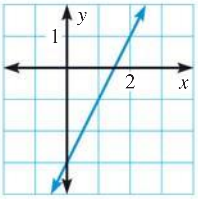
GIVEN ▶ A rotation about P maps Q to Q' and R to R'.
P and R are the same point.

PROVE ▶ $QR = Q'R'$

36. **MULTI-STEP PROBLEM** Use the graph of $y = 2x - 3$.

a. Rotate the line 90°, 180°, 270°, and 360° about the origin. *Describe* the relationship between the equation of the preimage and each image.

b. Do you think that the relationships you described in part (a) are true for *any* line? *Explain* your reasoning.

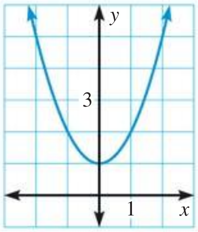


37. ★ **EXTENDED RESPONSE** Use the graph of the quadratic equation $y = x^2 + 1$ at the right.

a. Rotate the *parabola* by replacing y with x and x with y in the original equation, then graph this new equation.

b. What is the angle of rotation?

c. Are the image and the preimage both functions? *Explain.*

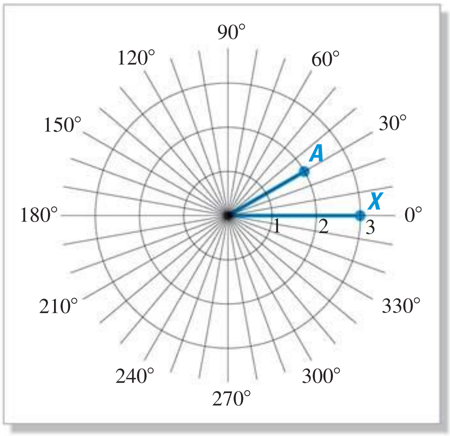


TWO ROTATIONS **The endpoints of $\overline{FG}$ are $F(1, 2)$ and $G(3, 4)$. Graph $\overline{F'G'}$ and $\overline{F''G''}$ after the given rotations.**

38. **Rotation:** 90° about the origin
Rotation: 180° about (0, 4)

39. **Rotation:** 270° about the origin
Rotation: 90° about (−2, 0)

40. **CHALLENGE** A polar coordinate system locates a point in a plane by its distance from the origin O and by the measure of an angle with its vertex at the origin. For example, the point $A(2, 30°)$ at the right is 2 units from the origin and $m\angle XOA = 30°$. What are the polar coordinates of the image of point A after a 90° rotation? 180° rotation? 270° rotation? *Explain.*



MIXED REVIEW

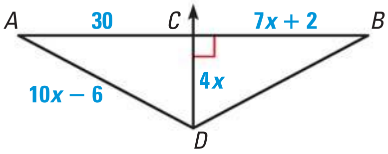
PREVIEW
Prepare for Lesson 9.5 in Exs. 41–43.

In the diagram, $\overrightarrow{DC}$ is the perpendicular bisector of $\overline{AB}$. *(p. 303)*

41. What segment lengths are equal?

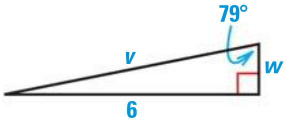
42. What is the value of x?

43. Find BD. *(p. 433)*

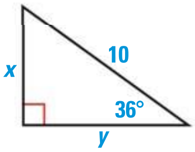


Use a sine or cosine ratio to find the value of each variable. Round decimals to the nearest tenth. *(p. 473)*

44.



45.



46.